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A CONCEPT OF NEGATIVE DEPENDENCE  
USING STOCHASTIC ORDERING

by

Henry W. Block<sup>1,3</sup>

Thomas H. Savits<sup>1,3</sup>

and

Moshe Shaked<sup>2,4</sup>

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Institute for Statistics and Applications  
Department of Mathematics and Statistics  
University of Pittsburgh  
Pittsburgh, PA 15260

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<sup>1</sup>Department of Mathematics and Statistics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260.

<sup>2</sup>Department of Mathematics, University of Arizona, Tucson, Arizona 85721.

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by

Henry W. Block

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and

Moshe Shaked



ABSTRACT

A concept of negative dependence called negative dependence by stochastic ordering is introduced. This concept satisfies various closure properties. It is shown that three models for negative dependence satisfy it and that it implies the basic negative orthant inequalities. This concept is also satisfied by the multinomial, multivariate hypergeometric, Dirichlet and Dirichlet compound multinomial distributions. Furthermore, the joint distribution of ranks of a sample and the multivariate normal with nonpositive pairwise correlations also satisfy this condition. The positive dependence analog of this condition is also studied.

## 1. Motivation and Preliminaries

In Block, Savits and Shaked (1982) an intuitive concept of negative independence is introduced. Essentially it requires that the sum of components for a random vector be constant. This condition is naturally satisfied by the multinomial, multivariate hypergeometric, Dirichlet, and Dirichelet compound multinomial distributions. Furthermore certain multivariate normal distributions with pairwise negative correlations (e.g. the symmetric ones) can be seen to satisfy this property, but it is not a natural one for the multivariate normal. In this paper we introduced a concept of negative dependence called negative dependence by stochastic ordering which is natural for the multivariate normal. It is easily shown that every multivariate normal whose components are pairwise negatively correlated satisfies this condition. Furthermore all of the other distributions mentioned above satisfy this condition. Finally three intuitive models are seen to satisfy this condition.

This new condition of negative dependence also satisfies the basic intuitive requirement for negative dependence that if a set of negatively dependent random variables is split into two subsets in some manner then one subset will tend to be large when the other subset is small. See also Alam and Lal Saxena (1981) and Jogdeo and Proschan (1981) and references there.

A random vector  $\underline{X}$  is said to be stochastically decreasing in the random vector  $\underline{Y}$  (notation:  $\underline{X} \downarrow \underline{Y}$ ) if the conditional expectation  $E[g(\underline{X}) | \underline{Y} = \underline{y}]$  is nonincreasing in  $\underline{y}$  whenever  $g$  is a nondecreasing Borel measurable function such that the above conditional expectations exist.

The basic concept of negative dependence to be discussed here is defined by requiring a random vector  $\underline{T}$  to satisfy

$$(1.1) \quad (T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_n) \stackrel{st}{\prec} T_i, \quad i=1, \dots, n.$$

As will be seen in Sections 3 and 4 our condition is often easily verifiable, it arises naturally in many applications, it implies the orthant inequalities

$$(1.2.a) \quad P(T_1 \leq t_1, \dots, T_n \leq t_n) \leq \prod_{i=1}^n P(T_i \leq t_i),$$

$$(1.2.b) \quad P(T_1 > t_1, \dots, T_n > t_n) \leq \prod_{i=1}^n P(T_i > t_i),$$

and it enjoys some closure properties which enable us to derive the inequalities (1.2) for many well known distributions.

The use of the modern theory of stochastic ordering will throw a new light on the underlying ideas of Mallows (1968) and Jogdeo and Patil (1975) who derived inequalities (1.2) for some well known distributions.

The positive dependence analog of (1.1) is also of some interest. It will be discussed in Section 5.

In the following "increasing" stands for "nondecreasing" and "decreasing" for "nonincreasing". Vectors in  $\mathbb{R}^n$  are denoted by  $\underline{t} = (t_1, \dots, t_n)$  and  $\underline{t} \leq \underline{t}'$  means  $t_i \leq t'_i$ ,  $i=1, \dots, n$ . Similarly  $\underline{t} < \underline{t}'$  means  $t_i < t'_i$ ,  $i=1, \dots, n$ , and  $\underline{0} = (0, \dots, 0)$ . A real function on  $\mathbb{R}^n$  will be called increasing if it is increasing in each variable when the other variables are held fixed. Whenever we write an expectation we assume that it exists and whenever an expectation or a probability is conditioned on an event such as  $\{\underline{Y} = \underline{y}\}$  we assume that  $\underline{y}$  is in the support of  $\underline{Y}$ .

The following definitions and results from the theory of stochastic ordering will be used.

A random variable  $X$  is said to be stochastically smaller than the random variable  $Y$  (denoted by  $X \leq_{st} Y$ ) if  $P(X > x) \leq P(Y > x)$  for every real  $x$ . The random vector  $\underline{X} = (X_1, \dots, X_n)$  is said to be stochastically smaller than  $\underline{Y} = (Y_1, \dots, Y_n)$  [denoted by  $\underline{X} \leq_{st} \underline{Y}$ ] if  $g(\underline{X}) \leq g(\underline{Y})$  for every  $g \in \mathcal{C}$  where  $\mathcal{C}$  is the class of Borel measurable increasing functions on  $\mathbb{R}^n$ . If  $\underline{X}$  and  $\underline{Y}$  have the same distribution then we write  $\underline{X} =_{st} \underline{Y}$ . It is well known that  $\underline{X} \leq_{st} \underline{Y}$  if and only if

$$P(\underline{X} \in U) \leq P(\underline{Y} \in U) \quad \text{for every upper Borel set } U \text{ in } \mathbb{R}^n.$$

( $U$  is an upper set if  $\underline{x} \in U$  and  $\underline{x} \leq \underline{y}$  implies that  $\underline{y} \in U$ .) According to Kamae, Krengel and O'Brien (1977), we need only consider open upper sets  $U$  in  $\mathbb{R}^n$ . It is also well known that if  $(X_1, \dots, X_n) \leq_{st} (Y_1, \dots, Y_n)$  then for any subcollection  $1 \leq i_1 < \dots < i_k \leq n$ ,

$$(1.3) \quad (X_{i_1}, \dots, X_{i_k}) \leq_{st} (Y_{i_1}, \dots, Y_{i_k}).$$

Also, if  $X_1 \leq_{st} Y_1, \dots, X_n \leq_{st} Y_n$  and if  $X_1, \dots, X_n$  are independent and  $Y_1, \dots, Y_n$  are independent, then

$$(1.4) \quad (X_1, \dots, X_n) \leq_{st} (Y_1, \dots, Y_n).$$

**Definition 1.1.** The random variables  $T_1, \dots, T_n$  for the random vector  $\underline{T}$  (or its distribution) are said to be negatively dependent through stochastic ordering (NDS) if (1.1) holds.

Note that NDS implies both NUOD and NLOD and these implications are sharp. To see this, use methods similar to Barlow and Proschan (1975), p. 143, and property (1.3) to show that (1.1) implies for  $i = 1, \dots, n-1$ ,

$$(1.5) \quad [(T_1, \dots, T_i) \mid T_{i+1} > t_{i+1}] \stackrel{st}{\geq} [(T_1, \dots, T_i) \mid T_{i+1} > t'_{i+1}]$$

whenever  $t_{i+1} \leq t'_{i+1}$ . (Although Barlow and Proschan assumed the existence of a density, a modification of their proof works in the general case.)

But from (1.5) it follows that

$$\begin{aligned} P(T_1 > t_1, \dots, T_n > t_n) &\leq P(T_1 > t_1, \dots, T_{n-1} > t_{n-1}) P(T_n > t_n) \\ &\leq P(T_1 > t_1, \dots, T_{n-2} > t_{n-2}) \prod_{i=n-1}^n P(T_i > t_i) \\ &\leq \dots \leq \prod_{i=1}^n P(T_i > t_i) \end{aligned}$$

which proves (1.2b). The proof of (1.1)  $\Rightarrow$  (1.2.a) is similar.

To justify calling (1.1) a "condition for negative dependence" we have to show that it implies

$$(1.6) \quad \text{COV}(T_i, T_j) \leq 0, \quad 1 \leq i < j \leq n,$$

when the second moments exist. From (1.2.a) it follows that

$P(T_i > t_i, T_j > t_j) \leq P(T_i > t_i) P(T_j > t_j)$  and it is well known that this inequality implies (1.6) [see e.g., Lehmann (1966)].

## 2. Closure Results

Preservation theorems are useful for identifying negatively dependent distributions or for constructing new negatively dependent distributions from known ones. In this section we discuss some preservation results. Their use will be illustrated in Sections 3 and 4.

**Theorem 2.1.** If  $T_1, \dots, T_n$  are NDS and if  $\psi_1, \dots, \psi_n$  are strictly increasing functions then  $\psi_1(T_1), \dots, \psi_n(T_n)$  are NDS.

Theorem 2.2. If  $(T_1, \dots, T_n)$  and  $(S_1, \dots, S_n)$  are independent and are NDS then  $(T_1, \dots, T_n, S_1, \dots, S_n)$  is NDS.

The proofs of these theorems are straightforward and will be omitted.

The following preliminaries are needed for the statement of Theorem 2.3; a thorough discussion can be found in Karlin (1968). A bivariate function  $K(\cdot, \cdot)$  which is defined on  $S_1 \times S_2$  (where  $S_1$  and  $S_2$  are subsets of  $\mathbb{R}$ ) is said to be totally positive of order 2 ( $TP_2$ ) on  $S_1 \times S_2$  if  $K(x, y) \geq 0$  and if

$$K(x, y) K(x', y') \geq K(x, y') K(x', y) \text{ whenever } x \leq x', y \leq y'.$$

A univariate density  $f$  is said to be a Polya frequency function of order 2 ( $PF_2$ ) if  $f(x-y)$  is  $TP_2$  on  $\mathbb{R} \times \mathbb{R}$ . Equivalently,  $f$  is  $PF_2$  if it is log concave. A probability function  $f$  is  $PF_2$  if  $f(x-y)$  is  $TP_2$  on  $N \times N$  where  $N = \{\dots, -1, 0, 1, \dots\}$ .

Theorem 2.3. Assume that  $(T_1, \dots, T_n)$  and  $(S_1, \dots, S_n)$  are independent and NDS. If all the univariate marginal densities (with respect to Lebesgue measure), or probability functions in the discrete case, of  $\underline{S}$  and  $\underline{T}$  are  $PF_2$ , then  $(T_1 + S_1, \dots, T_n + S_n)$  is NDS.

Remark. Karlin and Rinott (1980) have introduced a condition of negative dependence and have proven a related result for their condition. They assumed that  $\underline{S}$  and  $\underline{T}$  satisfy their condition and that they have  $PF_2$  marginals, and they showed that then  $\underline{S} + \underline{T}$  satisfy some inequalities that are stronger versions of (1.2.a) and (1.2.b).

The proof of Theorem 2.3 is easily obtained from the following lemmas.

Lemma 2.1. Let  $X_1, \dots, X_n$  be independent random variables with  $PF_2$  densities or probability functions. Then



$$(2.1) \quad (X_1, \dots, X_n) \overset{\text{st}}{\downarrow} X_1 + \dots + X_n.$$

Proof. See Efron (1965).

Lemma 2.2. Let  $\underline{X} = (X_1, \dots, X_n)$  and  $\underline{Y} = (Y_1, \dots, Y_n)$  be independent and assume

$$(2.2) \quad (X_1, \dots, X_{n-1}) \overset{\text{st}}{\downarrow} X_n$$

and

$$(2.3) \quad (Y_1, \dots, Y_{n-1}) \overset{\text{st}}{\downarrow} Y_n.$$

Furthermore assume that  $X_n$  and  $Y_n$  have  $PF_2$  densities or probability functions. Then

$$(X_1 + Y_1, \dots, X_{n-1} + Y_{n-1}) \overset{\text{st}}{\downarrow} X_n + Y_n.$$

Proof: Clearly, for any increasing function  $g$ ,

$$E[g(X_1 + Y_1, \dots, X_{n-1} + Y_{n-1}) \mid X_n + Y_n = z_n] = E[\phi(X_n, Y_n) \mid X_n + Y_n = z_n]$$

where  $\phi(x_n, y_n) = E[g(X_1 + Y_1, \dots, X_{n-1} + Y_{n-1}) \mid X_n = x_n, Y_n = y_n]$ . However,

$\phi(x_n, y_n)$  decreases in  $x_n$  and in  $y_n$  because of (2.2), (2.3) and independence.

Thus, by Lemma 2.1,  $E[\phi(X_n, Y_n) \mid X_n + Y_n = z_n]$  decreases in  $z_n$ . ||

### 3. Models

There are various models which give rise to the NDS condition. The standard negatively dependent distributions such as the Dirichlet, multinomial, multivariate hypergeometric, multivariate negative binomial and various negatively correlated normals are easily seen (below) to be generated by such models.

Model 1 If  $X_0, X_1, \dots, X_n$  are independent random variables with  $PF_2$  densities or probability functions then the random vector  $(T_1, \dots, T_n)$  which has the distribution determined by the equation

$$(T_1, \dots, T_n) \stackrel{\text{st}}{=} [(X_1, \dots, X_n) | a_0 X_0 + a_1 X_1 + \dots + a_n X_n = z]$$

for some constants  $a_i > 0$ ,  $i = 0, \dots, n$ , and  $z$  is NDS.

Proof. Let  $g$  be an  $(n-1)$ -variate increasing function and let  $t_n$  be a point in the support of  $T_n$ . Then using the fact that  $(X_1, \dots, X_{n-1})$  and  $X_n$  are independent,

$$\begin{aligned} b(t_n) &= E[g(T_1, \dots, T_{n-1}) | T_n = t_n] \\ &= E[g(X_1, \dots, X_{n-1}) | X_n = t_n, a_0 X_0 + \dots + a_{n-1} X_{n-1} + a_n X_n = z] \\ &= E[g(X_1, \dots, X_{n-1}) | a_0 X_0 + \dots + a_{n-1} X_{n-1} = z - a_n t_n]. \end{aligned}$$

Let  $Z_i = a_i X_i$ ,  $i = 0, 1, \dots, n-1$  and note that  $Z_0, Z_1, \dots, Z_{n-1}$  are independent with  $PF_2$  densities or probability functions. Define the function  $\tilde{g}$  by  $\tilde{g}(x_1, \dots, x_n) = g(\frac{x_1}{a_1}, \dots, \frac{x_{n-1}}{a_{n-1}})$ . Then

$$b(t_n) = E[\tilde{g}(Z_1, \dots, Z_{n-1}) | Z_0 + Z_1 + \dots + Z_{n-1} = z - a_n t_n].$$

By Lemma 2.1,  $b(t_n)$  decreases in  $t_n$  because  $a_n > 0$ . That is,  $(T_1, \dots, T_{n-1}) \stackrel{\text{st}}{\downarrow} T_n$ . The other conditions of (1.1) can be shown similarly. ||

Model 1 is essentially equivalent to the structure condition of Block, Savits and Shaked (1982). Various other properties that are enjoyed by this model are discussed there.

Model 2 If  $X_1, \dots, X_n$  are independent, identically distributed random variables having either a continuous or a discrete distribution function  $F$ , then the random vector  $(T_1, \dots, T_n)$ , which has the distribution determined by the equation

$$(3.1.a) \quad (T_1, \dots, T_n) \stackrel{\text{st}}{=} [(X_1, \dots, X_n) | \min(X_1, \dots, X_n) = z]$$

or by the equation

$$(3.1.b) \quad (T_1, \dots, T_n) \stackrel{\text{st}}{=} [(X_1, \dots, X_n) | \max(X_1, \dots, X_n) = z]$$

for some constant  $z$ , is NDS.

We will prove the above only for the case where  $F$  is assumed to be continuous. The discrete case is handled similarly.

The following lemma is directly verifiable.

Lemma 3.1. Let  $X_1, \dots, X_n$  be independent, identically distributed random variables with continuous distribution  $F$ . Then for  $k = 0, 1, \dots, n$  and  $z$  in the support of  $F$ ,

$$(3.2) \quad P(X_1 > x_1, \dots, X_n > x_n | \min_{1 \leq i \leq n} X_i = z) = \frac{n-k}{n} \frac{1}{\bar{F}^k(z)} \prod_{j=1}^k \bar{F}(x^{(n-j+1)})$$

for  $x^{(n-k)} < z \leq x^{(n-k+1)}$ , where  $x^{(1)} \leq x^{(2)} \leq \dots \leq x^{(n)}$  are the ordered  $x_i$ 's and we set  $x^{(0)} = -\infty$ ,  $x^{(n+1)} = +\infty$ , and the right hand side of (3.2) equal to zero for  $k = 0$ .

Notation. We will sometimes find it convenient to denote the right-hand side of (3.2) by  $\phi_n(\underline{x}; z)$ .

Corollary. For the  $X_1, \dots, X_n$  as above, we have that

$$(X_1, \dots, X_n) \stackrel{\text{st}}{\uparrow} \min_{1 \leq i \leq n} X_i.$$

Proof. We first claim that if  $g$  is any bounded Borel measurable function, then

$$E[g(X_1, \dots, X_n) | \min_{1 \leq i \leq n} X_i = z] = \frac{1}{n} \sum_{i=1}^n b_i(z),$$

where

$$b_i(z) = \int_0^\infty \dots \int_0^\infty g(y_1, \dots, y_{i-1}, z, y_{i+1}, \dots, y_n) \prod_{j \neq i} dF(y_j | z)$$

and  $\bar{F}(y|z) = \bar{F}(y)/\bar{F}(z)$  for  $y \geq z$  and one otherwise. Since this is true for indicator functions  $g$  of sets of the form  $(x_1, \infty) \times \dots \times (x_n, \infty)$  by (3.2), the claim follows. Thus to prove our result we need only show that each  $b_i$  is increasing in  $z$  whenever  $g$  is an increasing function. But since  $\bar{F}(y|z_1) \leq \bar{F}(y|z_2)$  if  $z_1 \leq z_2$  ( $z_i$  in the support of  $F$ ), this fact readily follows.

The next lemma is easy to prove.

Lemma 3.2. Let  $X$  be any random variable and  $Z$  be any random variable for which  $E[Z|X]$  is defined. If  $E[Z|X=x] = \phi(x)$ , then

$$E[Z|X \wedge a] = \begin{cases} \phi(X) & \text{if } X < a \\ c(a) & \text{if } X = a \end{cases}$$

where  $c(a) = E[\phi(X); X \geq a]/P(X \geq a)$  if  $P(X \geq a) > 0$  and zero otherwise.

Corollary. Suppose that  $E[Z|X=x] = \phi(x)$  is increasing in  $x$ . Then if  $P[X \geq x] > 0$ ,  $E[Z|X \wedge a = x]$  is decreasing in  $a \geq x$ .

Proof. Let  $P[X \geq x] > 0$  and suppose  $a_2 > a_1 \geq x$ . Then

$$E[Z|X \wedge a_1 = x] = \begin{cases} \phi(x) & \text{if } a_1 > x \\ c(x) & \text{if } a_1 = x \end{cases}$$

while

$$E[Z|X \wedge a_2 = x] = \phi(x)$$

according to Lemma 3.2. Hence we need only show that  $\phi(x) \leq c(x)$ .

But

$$c(x) = \frac{1}{P[X \geq x]} E[\phi(X) : X \geq x] \geq \phi(x)$$

since  $\phi$  is increasing.

Lemma 3.3. Let  $X_1, \dots, X_n$  be independent, identically distributed random variables with continuous distribution function  $F$ . Suppose that for some

$z$ ,  $(Y_1, \dots, Y_n) \stackrel{\text{st}}{=} [(X_1, \dots, X_n) | \min_{1 \leq i \leq n} X_i = z]$ . Then

$$[(Y_1, \dots, Y_{n-1}) | Y_n = w] \stackrel{\text{st}}{=} [(X_1, \dots, X_{n-1}) | (\min_{1 \leq i \leq n-1} X_i) \wedge w = z].$$

Proof. Let  $\underline{y} = (y_1, \dots, y_{n-1})$  and  $y_n$  be given. Then according to Lemmas 3.1 and 3.2, we have

$$P(Y_1 > y_1, \dots, Y_{n-1} > y_{n-1}, Y_n > y_n) = \phi_n(\underline{y}, y_n; z),$$

where  $\phi_n(\underline{y}, y_n; z)$  is the function given by the right hand side of (3.2), and

$$(3.3) \quad P(X_1 > y_1, \dots, X_{n-1} > y_{n-1} | \min_{1 \leq i \leq n-1} (X_i) \wedge y_n = z) = \begin{cases} \phi_{n-1}(\underline{y}; z) & \text{if } z < y_n \\ c(y_n) & \text{if } z = y_n. \end{cases}$$

In this case,

$$c(y_n) = \frac{P(X_1 > y_1, \dots, X_{n-1} > y_{n-1}; X_1 \geq y_n, \dots, X_{n-1} \geq y_n)}{P(X_1 \geq y_n, \dots, X_{n-1} \geq y_n)} = \frac{\bar{F}(y_1 \vee y_n) \dots \bar{F}(y_{n-1} \vee y_n)}{\bar{F}^{n-1}(y_n)}.$$

So if we denote the left-hand side of (3.3) by  $\psi(y_n)$ , we need only show that for all  $y_n$ ,

$$(3.4) \quad E[\psi(Y_n); Y_n > y_n] = \phi_n((y, y_n); z)$$

Using the fact that

$$P(Y_n > t) = \begin{cases} 1 & \text{if } t < z \\ \frac{n-1}{n} \frac{\bar{F}(t)}{\bar{F}(z)} & \text{if } t \geq z, \end{cases}$$

which follows from (3.2), the validity of (3.4) follows readily.

Proof of claim of Model 2. We will show that if  $\underline{T}$  has the representation (3.1.a), then  $(T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_n) \stackrel{\text{st}}{=} T_i$  for  $i = n$ . The other cases follow similarly. But this is a direct consequence of Lemma 3.3 and the corollaries following Lemmas 3.1 and 3.2. Similarly one shows that if  $\underline{T}$  has the representation (3.1.b), then  $\underline{T}$  is NDS.

Model 3 A model (suggested to us by Steve Arnold) which arises quite frequently in statistics is defined by the equation

$$(3.5) \quad (T_1, \dots, T_n) \stackrel{\text{st}}{=} (X_1 - \bar{X}, \dots, X_n - \bar{X})$$

where  $X_1, \dots, X_n$  are independent, identically distributed random variables and  $\bar{X} = n^{-1} \sum_{i=1}^n X_i$ . It is easily seen that if  $X_i$  is normal,  $i = 1, \dots, n$ , then  $(T_1, \dots, T_n)$  is multivariate normal with negative correlations and by Block, Savits and Shaked (1982) it satisfies the structure of Model 1. It is tempting to conjecture that a similar result holds for other random samples since the correlations are always negative. However if  $\underline{X} = (X_1, X_2, X_3)$  and if  $X_i$  can take on only two values, 0 and 1, and if  $P(X_i = 0)$  is close to 0,  $i = 1, 2, 3$ , then the random vector  $(T_1, T_2, T_3)$  of (3.5) need not even satisfy (1.2.a) or (1.2.b).

#### 4. Examples

The multinomial, the multivariate hypergeometric, the Dirichlet and the Dirichlet compound multinomial distributions are NDS because they are of the form of Model 1, as is shown in Block, Savits and Shaked (1982). Some other examples are listed below.

Example 4.1. The multivariate normal distribution with nonpositive correlations is NDS as can be verified by writing down explicitly the conditional distributions described in (1.1). This fact should be contrasted with results of Karlin and Rinott (1980) and Block, Savits and Shaked (1982) which show that some (but not all) multivariate normal distributions with nonpositive correlations satisfy various conditions of negative dependence.

Example 4.2. Mallows (1968) claimed, without proof, that convolutions of  $n$ -variate multinomials with possibly different sets of parameters satisfy (1.2.a) and (1.2.b). The fact that every multinomial distribution is NDS and Theorem 2.3 provide a proof of this claim. Clearly, similar results hold for the other NDS distributions.

Example 4.3. Jogdeo and Patil (1975) showed that the joint distribution of the ranks of any sample from a continuous distribution satisfies (1.2.a) and (1.2.b). It is not hard to show that this distribution is actually NDS. This example can be used to show that the class of NDS distributions contains some important distributions which do not satisfy the " $RR_2$  in pairs" condition of Block, Savits and Shaked (1982) and the  $S-MRR_2$  condition of Karlin and Rinott (1980).

### 5. A Positive Dependence Analog

We will say that the random variables  $T_1, \dots, T_n$  (or the random vector  $\underline{T}$  or its distribution) are positively dependent through stochastic ordering (PDS) if

$$(5.1) \quad (T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_n) \overset{\text{st}}{\uparrow} T_i, \quad i=1, \dots, n.$$

Although a multitude of concepts of positive dependence has been discussed recently (see, e.g., Barlow and Proschan (1975), Ch. 5, Shaked (1977, 1982) and Block and Ting (1981) and references there) it is somewhat surprising that (5.1) has not received any attention in the literature. In this section we will briefly mention some properties of this concept and where it stands in relation to other well known concepts.

Theorem 5.1. If  $\underline{T}$  is PDS then

$$(5.2.a) \quad P(T_1 \leq t_1, \dots, T_n \leq t_n) \geq \prod_{i=1}^n P(T_i \leq t_i),$$

$$(5.2.b) \quad P(T_1 > t_1, \dots, T_n > t_n) \geq \prod_{i=1}^n P(T_i > t_i).$$

The proof of this result is similar to the proof that the inequalities (1.2.a) and (1.2.b) follow from (1.1). The details are omitted.

Let  $I$  and  $I'$  be two intervals on the real line and denote  $I < I'$  whenever  $x \in I, y \in I'$  implies  $x < y$ . If  $\underline{T}$  is PDS and  $g$  is an increasing  $(n-1)$ -variable function then

$$(5.3) \quad \begin{aligned} & E g(T_1, \dots, T_{n-1}) \chi_I(T_n) E \chi_{I'}(T_n) \\ & \leq E g(T_1, \dots, T_{n-1}) \chi_{I'}(T_n) E \chi_I(T_n) \end{aligned}$$



whenever  $I < I'$  where  $\chi_A$  denote the indicator function of  $A$ . Inequalities such as (5.3) are sometimes useful. Using them, an alternative proof of Theorem 5.1 is obtained.

Theorem 5.2. Assume that  $(T_1, \dots, T_n)$  and  $(S_1, \dots, S_n)$  are independent and PDS. If all the univariate marginal densities, or probability functions of  $\underline{T}$  and  $\underline{S}$  are  $PF_2$  then  $(T_1 + S_1, \dots, T_n + S_n)$  is PDS.

The proof of this theorem is similar to the proof of Theorem 2.3. We mention that positive dependence analogs of Theorems 2.1 and 2.2 can also be stated and easily proven.

Theorem 5.3. Assume that  $\underline{T}$  satisfies  $T_{\pi(i+1)} \overset{st}{\uparrow} (T_{\pi(1)}, \dots, T_{\pi(i)})$  for  $i = 1, \dots, n-1$  for every permutation  $\pi$  of  $\{1, \dots, n\}$  [in other words, assume that  $T_{\pi(1)}, \dots, T_{\pi(n)}$  satisfy the CIS condition of Barlow and Proschan (1975), Ch. 5, for every permutation  $\pi$ ]. Then  $\underline{T}$  is PDS.

Proof. Let  $g$  be an increasing  $(n-1)$ -variate function. Since  $T_1 \overset{st}{\uparrow} (T_2, T_3, \dots, T_n)$  it follows that

$$E[g(T_1, t_2, t_3, \dots, t_{n-1}) | T_2 = t_2, \dots, T_{n-1} = t_{n-1}, T_n = t_n] \uparrow t_2, \dots, t_n.$$

Since  $T_2 \overset{st}{\uparrow} (T_3, \dots, T_n)$  it follows that

$$\begin{aligned} & E[g(T_1, T_2, t_3, \dots, t_{n-1}) | T_3 = t_3, \dots, T_{n-1} = t_{n-1}, T_n = t_n] \\ &= E[E[g(T_1, T_2, t_3, \dots, t_{n-1}) | T_2, T_3 = t_3, \dots, T_{n-1} = t_{n-1}, T_n = t_n] \\ & \quad | T_3 = t_3, \dots, T_{n-1} = t_{n-1}, T_n = t_n] \uparrow t_3, \dots, t_n. \end{aligned}$$

Continuing this way one obtains that

$$E[g(T_1, \dots, T_{n-1}) | T_n = t_n] \uparrow t_n$$

which proves (5.1) for  $i=n$ . The proof for  $i=1, \dots, n-1$  is similar. ||

It is well known that the CIS condition of Barlow and Proschan (1975) implies the association condition of Esary, Proschan and Walkup (1967). Association implies various positive dependence conditions of Shaked (1982) which in turn imply (5.2.a) and (5.2.b). The following example shows that the PDS condition does not imply any of the above except (5.2.a) and (5.2.b).

Example 5.1. Let  $\underline{X} = (X_1, X_2, X_3, X_4)$  be a vector of binary random variables with probabilities  $P(\underline{X} = (0,0,0,0)) = 4/24$ ,  $P(\underline{X} = (0,1,1,1)) = P(\underline{X} = (1,0,1,1)) = P(\underline{X} = (1,1,0,1)) = P(\underline{X} = (1,1,1,0)) = P(\underline{X} = (1,1,1,1)) = 2/24$  and the probabilities of any of the other ten outcomes are all  $1/24$ . Tedious computation shows that  $\underline{X}$  is PDS. However  $\text{Cov}(\min(X_1, X_2), \min(X_3, X_4)) = -1/576$  hence  $X_1, \dots, X_4$  are not associated and do not belong to the family  $\text{FPD}(F_1)$  of Shaked (1982). Since, in this example,  $\text{Cov}(\min(X_1, X_2), \min(X_3, X_4)) = P(X_1 + X_2 > 1.5, X_3 + X_4 > 1.5) - P(X_1 + X_2 > 1.5) P(X_3 + X_4 > 1.5)$  it also follows that  $\underline{X}$  does not belong to the family  $\text{PD}(A_2)$  of Shaked (1982). Thus,  $\underline{X}$  does not belong to any of the families of Shaked (1982) except for the family  $\text{FPD}(F_2)$ .

Example 5.2. Let  $\underline{X}$  be a multivariate normal random vector with nonnegative correlations. By explicitly writing the conditional distributions described in (5.1) it is easy to see that  $\underline{X}$  is PDS. This should be contrasted with the fact that  $\underline{X}$  need not be CIS. Recently it was proven by Pitt (1982) that  $\underline{X}$  above is associated.

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